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On the Affine Schur Algebra of Type A II

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Abstract

By studying certain kind of centralizer algebras of the affine Schur algebra $\widetilde{S}(n,r)$ we show that $\widetilde{S}(n,r)$ is Noetherian and we determine its center. Assuming $n \geq r$, we show that $\widetilde{S}(n+1,r)$ is Morita equivalent to $\widetilde{S}(n,r)$, and the Schur functor is an equivalence under certain conditions.

Key words : affine Schur algebra, centralizer algebra, Morita equivalence.

MSC2000 : 13F99, 16G30.

1 Introduction

The affine (q-)Schur algebra (of type A) has been studied by [1] [3] [4] [5] [6] [7], which provide various equivalent definitions of the algebra.

In this paper we define the affine Schur algebra $\widetilde{S}(n,r)$ $(n,r \in \mathbb{N})$ by giving a basis and the structure constants. We investigate certain centralizer algebras of $\widetilde{S}(n,r)$ of the form $e\widetilde{S}(n,r)e$, where e is an idempotent of $\widetilde{S}(n,r)$. The most interesting ones are those $\widetilde{S}(\underline{i})\xi_{\underline{i},\underline{i}}\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}$ for $\underline{i} \in I(n,r)$, and those isomorphic to $\widetilde{S}(n',r)$ for some $n' \in \mathbb{N}$. As applications, we obtain the following results. Firstly, when $n \geq r$, $\widetilde{S}(n+1,r)$ is Morita equivalent to $\widetilde{S}(n,r)$ (Theorem 3.4). Secondly, when $n \geq r$, the Schur functor is well-defined and it is an equivalence — thus the affine Schur algebra is Morita equivalent to the group algebra of the extended affine Weyl group — provided the characteristic of the base field is 0 or greater than r (Theorem 3.12). These two equivalences are affine analogues to the results in the finite case (*cf.* [2]). Thirdly, $\widetilde{S}(n,r)$ is Noetherian (Theorem 3.10). Besides, we determine the center of $\widetilde{S}(n,r)$. More precisely, its center is isomorphic to the algebra $K[t_1,\cdots,t_{r-1},t_r,t_r^{-1}]$ where $t_1,\cdots,t_r$ are indeterminates (Theorem 4.5).

This paper is organized as follows. In Section 2, we give the definition of the affine Schur algebra and recall some basic properties. Section 3 is devoted to the study of certain centralizer algebras. In this section we show the Morita equivalences stated above and prove that the affine Schur algebra is Noetherian. In Section 4 we determine the center. Section 5 provides some examples of the affine Schur algebra.

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2 The affine Schur algebra

First let us introduce the setting.

K will be an infinite field, and $n, r \in \mathbb{N}$. Let Σ_r denote the symmetric group on r letters and $\widehat{\Sigma}_r = \Sigma_r \ltimes \mathbb{Z}^r$ the extended affine Weyl group of type A_{r-1}.

For a set S, we denote by $I(S,r)$ the set $\{\underline{i} = (i_1, \cdots, i_r) \mid i_t \in S,\ t = 1, \cdots, r\}$ of all r-tuples of elements in S. We often omit the brackets and commas in the expression of $\underline{i} \in I(S,r)$ if it does not cause confusion. Then Σ_r acts on the right on $I(S,r)$ by place permutation. We will abbreviate $I(\{1, \cdots, n\}, r)$ to $I(n,r)$. Σ_n acts on the left on $I(n,r)$.

$\widehat{\Sigma}_r$ acts on $I(\mathbb{Z},r)$ on the right with Σ_r acting by place permutation and $\mathbb{Z}^r$ acting by shifting, i.e. $\underline{i}\varepsilon = \underline{i} + n\varepsilon$ for $\underline{i} \in I(\mathbb{Z},r)$ and $\varepsilon \in \mathbb{Z}^r$, and on $I(\mathbb{Z},r) \times I(\mathbb{Z},r)$ diagonally. This action depends on the number n. Note that a representative set for $I(n,r)/\Sigma_r$ is also a representative set for $I(\mathbb{Z},r)/\widehat{\Sigma}_r$. For $\underline{i} \in I(\mathbb{Z},r)$, set $\widehat{\Sigma}_{\underline{i}}$ (resp. $\Sigma_{\underline{i}}$) be the stabilizer group of $\underline{i}$ in $\widehat{\Sigma}_r$ (resp. Σ_r), and for $\underline{i}, \underline{j} \in I(\mathbb{Z},r)$, set $\widehat{\Sigma}_{\underline{i},\underline{j}} = \widehat{\Sigma}_{\underline{i}} \cap \widehat{\Sigma}_{\underline{j}}$ (resp. $\Sigma_{\underline{i},\underline{j}} = \Sigma_{\underline{i}} \cap \Sigma_{\underline{j}}$), and so on. Note that if $\underline{i} \in I(n,r)$ then $\widehat{\Sigma}_{\underline{i}} = \Sigma_{\underline{i}}$.

To each pair $(\underline{i}, \underline{j}) \in I(\mathbb{Z},r) \times I(\mathbb{Z},r)$, we associate an element $\xi_{\underline{i},\underline{j}}$ such that $\xi_{\underline{i},\underline{j}} = \xi_{\underline{k},\underline{l}}$ if and only if $(\underline{i}, \underline{j}) \sim_{\widehat{\Sigma}_r} (\underline{k}, \underline{l})$. *The affine Schur algebra* $\widetilde{S}(n,r)$ is defined to be the K-algebra with basis $\{\xi_{\underline{i},\underline{j}} | \underline{i}, \underline{j} \in I(\mathbb{Z},r)\}$ and multiplication given by

$$\xi_{\underline{i},\underline{j}} \xi_{\underline{k},\underline{l}} = \sum_{(\underline{p},\underline{q}) \in I(\mathbb{Z},r) \times I(\mathbb{Z},r)/\widehat{\Sigma}_r} Z(\underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{p}, \underline{q}) \xi_{\underline{p},\underline{q}}$$

where $Z(\underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{p}, \underline{q}) = \#\{\underline{s} \in I(\mathbb{Z},r) | (\underline{i}, \underline{j}) \sim_{\widehat{\Sigma}_r} (\underline{p}, \underline{s}), (\underline{s}, \underline{q}) \sim_{\widehat{\Sigma}_r} (\underline{k}, \underline{l})\}$. We have

Proposition 2.1. *([6]Proposition4.2)*

(i) $\xi_{\underline{i},\underline{j}} \xi_{\underline{k},\underline{l}} = 0$ *unless* $\underline{j} \sim_{\widehat{\Sigma}_r} \underline{k}$.

(ii) $\xi_{\underline{i},\underline{i}} \xi_{\underline{i},\underline{j}} = \xi_{\underline{i},\underline{j}} = \xi_{\underline{i},\underline{j}} \xi_{\underline{j},\underline{j}}$*, for* $\underline{i}, \underline{j} \in I(\mathbb{Z},r)$.

(iii) $\sum_{\underline{i} \in I(n,r)/\Sigma_r} \xi_{\underline{i},\underline{i}}$ *is a decomposition of the identity into orthogonal idempotents.*

We have another product formula, which is proved in the end of [6] Section4.

Proposition 2.2. *For* $\underline{i}, \underline{j}, \underline{l} \in I(\mathbb{Z},r)$*, we have*

$$\xi_{\underline{i},\underline{j}} \xi_{\underline{j},\underline{l}} = \sum_{\delta \in \widehat{\Sigma}_{\underline{j},\underline{l}} \backslash \widehat{\Sigma}_{\underline{j}} / \widehat{\Sigma}_{\underline{i},\underline{j}}} [\widehat{\Sigma}_{\underline{i},\underline{l}\delta} : \widehat{\Sigma}_{\underline{i},\underline{j},\underline{l}\delta}] \xi_{\underline{i},\underline{l}\delta} = \sum_{\delta \in \widehat{\Sigma}_{\underline{i},\underline{j}} \backslash \widehat{\Sigma}_{\underline{j}} / \widehat{\Sigma}_{\underline{j},\underline{l}}} [\widehat{\Sigma}_{\underline{i}\delta,\underline{l}} : \widehat{\Sigma}_{\underline{i}\delta,\underline{j},\underline{l}}] \xi_{\underline{i}\delta,\underline{l}} \ .$$

Let $^- : \mathbb{Z} \to \{1, \cdots, n\}$ be the map taking least positive remainder modulo n. It can be extended to $^- : I(\mathbb{Z},r) \to I(n,r)$. Note that $\xi_{\underline{i},\underline{j}} = \xi_{\bar{\underline{i}},\underline{j}+n\varepsilon^1} = \xi_{\underline{i}+n\varepsilon^2,\bar{\underline{j}}}$, where $\varepsilon^1 = \frac{\bar{\underline{i}}-\underline{i}}{n}$, $\varepsilon^2 = \frac{\bar{\underline{j}}-\underline{j}}{n}$ are both in $\mathbb{Z}^r$. Since $\{(\underline{i}, \underline{j} + n\varepsilon) \mid \underline{i} \in I(n,r)/\Sigma_r, \underline{j} \in I(n,r)/\Sigma_{\underline{i}}, \varepsilon \in \mathbb{Z}^r/\Sigma_{\underline{i},\underline{j}}\}$ is a set of representatives of the $\widehat{\Sigma}_r$-orbits of $I(\mathbb{Z},r) \times I(\mathbb{Z},r)$, the set $\{\xi_{\underline{i},\underline{j}} \mid \underline{i}, \underline{j} \in I(\mathbb{Z},r)\} = \{\xi_{\underline{i},\underline{j}} \mid (\underline{i}, \underline{j}) \in I(\mathbb{Z},r) \times I(\mathbb{Z},r)/\widehat{\Sigma}_r\}$ equals the set $\{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \underline{i}, \underline{j} \in I(n,r), \varepsilon \in \mathbb{Z}^r\} = \{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \underline{i} \in I(n,r)/\Sigma_r, \underline{j} \in I(n,r)/\Sigma_{\underline{i}}, \varepsilon \in \mathbb{Z}^r/\Sigma_{\underline{i},\underline{j}}\}$. Thus we can rewrite Proposition 2.2 as follows.

Proposition 2.3. *For* $\underline{i}, \underline{j}, \underline{l} \in I(n,r)$, $\varepsilon, \varepsilon' \in \mathbb{Z}^r$*, we have*

$$\begin{aligned}\xi_{\underline{i},\underline{j}+n\varepsilon} \xi_{\underline{j},\underline{l}+n\varepsilon'} &= \textstyle\sum_{\delta \in \Sigma_{\underline{j},\underline{l},\varepsilon'} \backslash \Sigma_{\underline{j}} / \Sigma_{\underline{i},\underline{j},\varepsilon}} [\Sigma_{\underline{i},\underline{l}\delta,\varepsilon'\delta+\varepsilon} : \Sigma_{\underline{i},\underline{j},\underline{l}\delta,\varepsilon'\delta,\varepsilon}] \xi_{\underline{i},\underline{l}\delta+n(\varepsilon'\delta+\varepsilon)} \\ &= \textstyle\sum_{\delta \in \Sigma_{\underline{i},\underline{j},\varepsilon} \backslash \Sigma_{\underline{j}} / \Sigma_{\underline{j},\underline{l},\varepsilon'}} [\Sigma_{\underline{i}\delta,\underline{l},\varepsilon'+\varepsilon\delta} : \Sigma_{\underline{i}\delta,\underline{j},\underline{l},\varepsilon',\varepsilon\delta}] \xi_{\underline{i}\delta,\underline{l}+n(\varepsilon'+\varepsilon\delta)} \ .\end{aligned}$$

3 Certain centralizer algebras

In this section we study centralizer algebras of the form $e\widetilde{S}(n,r)e$, where e is an idempotent of $\widetilde{S}(n,r)$. We show a few Morita equivalences and prove that $\widetilde{S}(n,r)$ is Noetherian.

Proposition 2.1(iii) says that $\sum_{\underline{i}\in I(n,r)/\Sigma_r}\xi_{\underline{i},\underline{i}}$ is a decomposition of the identity into orthogonal idempotents. Correspondingly $\oplus_{\underline{i}\in I(n,r)/\Sigma_r}\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}$ is a decomposition of the regular $\widetilde{S}(n,r)$-module into a direct sum of projective modules.

Lemma 3.1. *Let $\underline{i},\underline{j}\in I(n,r)$ with $\Sigma_{\underline{i}}\geq\Sigma_{\underline{j}}$. Assume char$K\dagger[\Sigma_{\underline{i}}:\Sigma_{\underline{j}}]$, then as an $\widetilde{S}(n,r)$-module $\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}$ is a direct summand of $\widetilde{S}(n,r)\xi_{\underline{j},\underline{j}}$.*

Proof. Note that under that assumption

$$\xi_{\underline{i},\underline{j}}\xi_{\underline{j},\underline{i}}=\sum_{\delta\in\Sigma_{\underline{i},\underline{j}}\backslash\Sigma_{\underline{j}}/\Sigma_{\underline{i},\underline{j}}}[\Sigma_{\underline{i},\underline{i}\delta}:\Sigma_{\underline{i},\underline{i}\delta,\underline{j}}]\xi_{\underline{i},\underline{i}\delta}=[\Sigma_{\underline{i}}:\Sigma_{\underline{j}}]\xi_{\underline{i},\underline{i}}$$

So $\phi:\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}\to\widetilde{S}(n,r)\xi_{\underline{j},\underline{j}}$, $\xi\mapsto\xi\xi_{\underline{i},\underline{j}}$, and $\psi:\widetilde{S}(n,r)\xi_{\underline{j},\underline{j}}\to\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}$, $\xi\mapsto\frac{\xi\xi_{\underline{j},\underline{i}}}{[\Sigma_{\underline{i}}:\Sigma_{\underline{j}}]}$ are both homomorphisms of $\widetilde{S}(n,r)$-modules and $\psi\circ\phi=id$. Consequently we get the desired result. □

For $\underline{i}\in I(n,r)$, denote by $\widetilde{S}(\underline{i})$ the centralizer algebra $\xi_{\underline{i},\underline{i}}\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}$. The following is a corollary of Lemma 3.1.

Lemma 3.2. *Let $\underline{i}\in I(n,r)$ and $\underline{j}=\sigma(\underline{i})$ for some $\sigma\in\Sigma_n$. Then $\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}$ and $\widetilde{S}(n,r)\xi_{\underline{j},\underline{j}}$ are isomorphic $\widetilde{S}(n,r)$-modules. Consequently, $\widetilde{S}(\underline{i})$ and $\widetilde{S}(\underline{j})$ are isomorphic K-algebras.*

Proof. The first statement follows from Lemma 3.1 since $\Sigma_{\underline{j}}=\Sigma_{\underline{i}}$. The second statement is because isomorphic modules have isomorphic endomorphism algebras. □

The next proposition shows that some centralizer algebras are in fact affine Schur algebras with different parameters.

Proposition 3.3. *Let N be a subset of $\{1,\cdots,n\}$, and $\xi_N=\sum_{\underline{i}\in I(N,r)/\Sigma_r}\xi_{\underline{i},\underline{i}}$. Then $\xi_N\widetilde{S}(n,r)\xi_N\cong\widetilde{S}(\#N,r)$.*

Proof. Let $s=\#N$, and $N^0=\{1,\cdots,s\}$. Then there exists a $\sigma\in\Sigma_n$ such that σ induces a bijection from $I(N,r)$ to $I(N^0,r)$ given by $\underline{i}\mapsto\sigma(\underline{i})$. Therefore by Lemma 3.2 $\widetilde{S}(n,r)\xi_N$ and $\widetilde{S}(n,r)\xi_{N^0}$ are isomorphic $\widetilde{S}(n,r)$-modules. As a result, their endomorphism algebras $\xi_N\widetilde{S}(n,r)\xi_N$ and $\xi_{N^0}\widetilde{S}(n,r)\xi_{N^0}$ are isomorphic K-algebras. Finally, sending $\xi_{\underline{i},\underline{j}+n\varepsilon}$ to $\xi_{\underline{i},\underline{j}+s\varepsilon}$ is a K-algebra isomorphism from $\xi_{N^0}\widetilde{S}(n,r)\xi_{N^0}$ to $\widetilde{S}(s,r)$. □

Now we are able to establish a Morita equivalence.

Theorem 3.4. *Assume $n\geq r$, then $\widetilde{S}(n+1,r)$ is Morita equivalent to $\widetilde{S}(n,r)$.*

Proof. Since $n\geq r$, we have that for any $\underline{i}\in I(n+1,r)$ there exists $\sigma\in\Sigma_{n+1}$ such that $\sigma(\underline{i})\in I(n,r)$. Therefore $\widetilde{S}(n+1,r)\xi_N$ is a progenerator where $N=\{1,\cdots,n\}$. In particular, $\widetilde{S}(n+1,r)$ is Morita equivalent to $\xi_N\widetilde{S}(n+1,r)\xi_N$, which is isomorphic to $\widetilde{S}(n,r)$ by Proposition 3.3. □

Next we will concentrate on the algebras $\widetilde{S}(\underline{i})$. We start with a special case.

3.1 Special case : $\widetilde{S}(1\cdots 1)$

$\widetilde{S}(1\cdots 1) = K\{\xi_{1\cdots 1,1\cdots 1+n\varepsilon} \mid \varepsilon \in \mathbb{Z}^r\}$. Following Proposition 2.3 we write down the product formula for this algebra

$$\begin{aligned}\xi_{1\cdots 1,1\cdots 1+n\varepsilon}\xi_{1\cdots 1,1\cdots 1+n\varepsilon'} &= \textstyle\sum_{\delta\in\Sigma_{\varepsilon'}\backslash\sigma_r/\Sigma_\varepsilon}[\Sigma_{\varepsilon'\delta+\varepsilon} : \Sigma_{\varepsilon'\delta,\varepsilon}]\xi_{1\cdots 1,1\cdots 1+n(\varepsilon'\delta+\varepsilon)}\\ &= \textstyle\sum_{\delta\in\Sigma_\varepsilon\backslash\sigma_r/\Sigma_{\varepsilon'}}[\Sigma_{\varepsilon\delta+\varepsilon'} : \Sigma_{\varepsilon\delta,\varepsilon'}]\xi_{1\cdots 1,1\cdots 1+n(\varepsilon\delta+\varepsilon')}\ ,\end{aligned}$$

where $\varepsilon, \varepsilon' \in \mathbb{Z}^r$. In particular $\widetilde{S}(1\cdots 1)$ is a commutative algebra. Since $\Sigma_{1\cdots 1} = \Sigma_r$, we see that $\xi_{1\cdots 1,1\cdots 1+n\varepsilon} = \xi_{1\cdots 1,1\cdots 1+n\varepsilon\sigma}$ for any $\varepsilon \in \mathbb{Z}^r$ and $\sigma \in \Sigma_r$. Thus for a basis element $\xi_{1\cdots 1,1\cdots 1+n\varepsilon}$ we can always choose ε to be weakly decreasing. In other words, $\xi_{1\cdots 1,1\cdots 1+n\varepsilon}$, $\varepsilon \in \mathbb{Z}^r$ weakly decreasing, form a basis for $\widetilde{S}(1\cdots 1)$.

We denote by $\widetilde{S}(1\cdots 1)^+$ the subalgebra of $\widetilde{S}(1\cdots 1)$ with basis $\{\xi_{1\cdots 1,1\cdots 1+n\varepsilon} \mid \varepsilon = (\varepsilon_1, \cdots, \varepsilon_r) \in \mathbb{Z}^r, \varepsilon_1 \geq \cdots \geq \varepsilon_r \geq 0\}$. Then $\widetilde{S}(1\cdots 1)^+$ is a positively graded K-algebra with grading given by $deg(\xi_{1\cdots 1,1\cdots 1+n(\varepsilon_1,\cdots,\varepsilon_r)}) = \varepsilon_1 + \cdots + \varepsilon_r$. Let $e^1, \cdots, e^r$ be the natural $\mathbb{Z}$-basis for $\mathbb{Z}^r$. Let $\varepsilon^k = \sum_{i=1}^k e^i$ and $t_k = \xi_{1\cdots 1,1\cdots 1+n\varepsilon^k}$ for $k = 1, \cdots, r$.

Proposition 3.5. $\widetilde{S}(1\cdots 1)^+ = K\xi_{1\cdots 1,1\cdots 1}[t_1, \cdots, t_{r-1}, t_r]$ *is a polynomial algebra in r indeterminates.*

Proof. We prove in two steps: $t_1, \cdots, t_r$ generate $\widetilde{S}(1\cdots 1)^+$ and they are algebraically independent. For the first step, it suffices to show basis elements $\xi_{1\cdots 1,1\cdots 1+n\varepsilon}$, $\varepsilon \in \mathbb{Z}^r$ weakly decreasing, are generated by them. Induct on the degree and the number of s's with $\varepsilon_s = 0$.

If $\varepsilon_r \neq 0$, then $\xi_{1\cdots 1,1\cdots 1+n\varepsilon} = \xi_{1\cdots 1,1\cdots 1+n(\varepsilon-\varepsilon_r\varepsilon^r)}t_r^{\varepsilon_r}$ is the product of an element of less degree and a power of t_r.

Assume $\varepsilon_{k+1} = \cdots = \varepsilon_r = 0$ but $\varepsilon_k \neq 0$, then

$$\xi_{1\cdots 1,1\cdots 1+n\varepsilon} = \xi_{1\cdots 1,1\cdots 1+n(\varepsilon-\varepsilon^k)}t_k - \sum_\delta [\Sigma_{\varepsilon^k\delta+\varepsilon-\varepsilon^k} : \Sigma_{\varepsilon^k\delta,\varepsilon-\varepsilon^k}]\xi_{1\cdots 1,1\cdots 1+n(\varepsilon^k\delta+\varepsilon-\varepsilon^k)}$$

where the sum is over a set of representatives of all non-trivial double cosets $\Sigma_{\varepsilon^k}\delta\Sigma_{\varepsilon-\varepsilon^k}$ of Σ_r. Note that $\xi_{1\cdots 1,1\cdots 1+n(\varepsilon-\varepsilon^k)}$ is of less degree and $\varepsilon^k\delta + \varepsilon - \varepsilon^k$ has less zero entries than ε. By induction we finish this step.

Now let us prove the second step. The K-dimension of the homogeneous component of $\widetilde{S}(1\cdots 1)^+$ of degree nm is $\#\{\varepsilon \in \mathbb{Z}^r | \varepsilon_1 \geq \cdots \geq \varepsilon_r \geq 0, \varepsilon_1 + \cdots + \varepsilon_r = m\}$. This set is in bijection with the set $\{(n_1, \cdots, n_r) \in \mathbb{Z}^r | n_1, \cdots, n_r \geq 0, n_1 + 2n_2 + \cdots + rn_r = m\}$, whose cardinality equals the K-dimension of the homogeneous component of the polynomial algebra $K[x_1, \cdots, x_r]$ of degree nm with $deg(x_k) = nk$, $k = 1, \cdots, r$. This dimension comparison shows that $t_1, \cdots, t_r$ are algebraically independent. $\square$

Let $\varepsilon \in \mathbb{Z}^r$ be weakly decreasing. If $\varepsilon_r < 0$, then $\xi_{1\cdots 1,1\cdots 1+n\varepsilon} = \xi_{1\cdots 1,1\cdots 1+n(\varepsilon-\varepsilon_r\varepsilon^r)}(t_r^{-1})^{-\varepsilon_r}$ is the product of an element $\xi_{1\cdots 1,1\cdots 1+n(\varepsilon-\varepsilon_r\varepsilon^r)}$ in $\widetilde{S}(1\cdots 1)^+$ and a power of t_r^{-1}, where $t_r^{-1} = \xi_{1\cdots 1,1\cdots 1+n(-\varepsilon^r)}$.

Proposition 3.6. (i) *As a K-algebra $\widetilde{S}(1\cdots 1)$ is isomorphic to $K[t_1, \cdots, t_{r-1}, t_r, t_r^{-1}]$, where $t_1, \cdots, t_r$ are indeterminates. In particular, the affine Schur algebra $\widetilde{S}(1, r)$ is isomorphic to $K[t_1, \cdots, t_{r-1}, t_r, t_r^{-1}]$.*

(ii) $\xi_{1\cdots 1,1\cdots 1}$ *is a primitive idempotent of* $\widetilde{S}(n,r)$.

3.2 A commutative subalgebra of $\widetilde{S}(\underline{i})$

Fix $\underline{i}\in I(n,r)$ in this subsection. Then $\widetilde{S}(\underline{i})=K\{\xi_{\underline{i},\underline{i}w}\mid w\in\widehat{\Sigma}_r\}=K\{\xi_{\underline{i},\underline{i}\sigma+n\varepsilon}\mid\sigma\in\Sigma_r,\varepsilon\in\mathbb{Z}^r\}$. The rest of this subsection is devoted to the study of a commutative subalgebra of $\widetilde{S}(\underline{i})$.

Let $B_{\underline{i}}=K\{\xi_{\underline{i},\underline{i}+n\varepsilon}\}$. By Proposition 2.3 for $\varepsilon,\varepsilon'\in\mathbb{Z}^r$,

$$\begin{aligned}\xi_{\underline{i},\underline{i}+n\varepsilon}\xi_{\underline{i},\underline{i}+n\varepsilon'}&=\textstyle\sum_{\delta\in\Sigma_{\underline{i},\varepsilon}\backslash\Sigma_{\underline{i}}/\Sigma_{\underline{i},\varepsilon'}}[\Sigma_{\underline{i},\varepsilon'+\varepsilon\delta}:\Sigma_{\underline{i},\varepsilon',\varepsilon\delta}]\xi_{\underline{i},\underline{i}+n(\varepsilon'+\varepsilon\delta)}\\&=\textstyle\sum_{\delta\in\Sigma_{\underline{i},\varepsilon'}\backslash\Sigma_{\underline{i}}/\Sigma_{\underline{i},\varepsilon}}[\Sigma_{\underline{i},\varepsilon+\varepsilon'\delta}:\Sigma_{\underline{i},\varepsilon,\varepsilon'\delta}]\xi_{\underline{i},\underline{i}+n(\varepsilon+\varepsilon'\delta)}\ .\end{aligned}$$

It follows that $B_{\underline{i}}$ is a commutative subalgebra of $\widetilde{S}(\underline{i})$. Moreover,

Proposition 3.7. *$B_{\underline{i}}$ is a maximal commutative subalgebra of $\widetilde{S}(\underline{i})$.*

Proof. Let $\xi=\sum_{\sigma\in\Sigma_{\underline{i}}\backslash\Sigma_r/\Sigma_{\underline{i}}}\sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{i}\sigma}}\lambda_{\sigma,\varepsilon}\xi_{\underline{i},\underline{i}\sigma+n\varepsilon}\in\widetilde{S}(\underline{i})$ be such that for any $\varepsilon^0\in\mathbb{Z}^r$,

$$\xi\xi_{\underline{i},\underline{i}+n\varepsilon^0}=\xi_{\underline{i},\underline{i}+n\varepsilon^0}\xi\ ,$$

$$\begin{aligned}\text{i.e.}\quad&\sum_{\sigma\in\Sigma_{\underline{i}}\backslash\Sigma_r/\Sigma_{\underline{i}}}\ \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{i}\sigma}}\lambda_{\sigma,\varepsilon}\sum_{\delta\in\Sigma_{\underline{i}\sigma,\varepsilon^0\sigma}\backslash\Sigma_{\underline{i}\sigma}/\Sigma_{\underline{i}\sigma,\underline{i},\varepsilon}}[\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon+\varepsilon^0\sigma\delta}:\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon,\varepsilon^0\sigma\delta}]\xi_{\underline{i},\underline{i}\sigma+n(\varepsilon+\varepsilon^0\sigma\delta)}\\=&\sum_{\sigma\in\Sigma_{\underline{i}}\backslash\Sigma_r/\Sigma_{\underline{i}}}\ \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{i}\sigma}}\lambda_{\sigma,\varepsilon}\sum_{\delta\in\Sigma_{\underline{i},\varepsilon^0}\backslash\Sigma_{\underline{i}}/\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon}}[\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon+\varepsilon^0\delta}:\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon,\varepsilon^0\delta}]\xi_{\underline{i},\underline{i}\sigma+n(\varepsilon+\varepsilon^0\delta)}\qquad(*)\end{aligned}$$

Take $t\in\mathbb{N}$ large enough, say, $t=10\times max\{|\varepsilon_s|\ \big|\ s=1,\cdots,r,\varepsilon:\lambda_{\sigma,\varepsilon}\neq0\ for\ some\ \sigma\}$ and take $\varepsilon^0=t(1,2,\cdots,r)$. Then for $\sigma\in\Sigma_{\underline{i}}\backslash\Sigma_r/\Sigma_{\underline{i}}$, $\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{i}\sigma}$, $\delta\in\Sigma_{\underline{i},\varepsilon^0}\backslash\Sigma_{\underline{i}}/\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon}$, we have $\Sigma_{\varepsilon+\varepsilon^0\delta}=1$. In particular, $[\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon+\varepsilon^0\delta}:\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon,\varepsilon^0\delta}]=1$. Moreover, for $\sigma'\in\Sigma_{\underline{i}}\backslash\Sigma_r/\Sigma_{\underline{i}}$, $\varepsilon'\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{i}\sigma'}$, $\delta'\in\Sigma_{\underline{i},\varepsilon^0}\backslash\Sigma_{\underline{i}}/\Sigma_{\underline{i},\underline{i}\sigma',\varepsilon'}$, we have $\xi_{\underline{i},\underline{i}\sigma+n(\varepsilon+\varepsilon^0\delta)}=\xi_{\underline{i},\underline{i}\sigma'+n(\varepsilon'+\varepsilon^0\delta')}$ implies $(\underline{i},\underline{i}\sigma+n(\varepsilon+\varepsilon^0\delta))\sim_{\widehat{\Sigma}_r}(\underline{i},\underline{i}\sigma'+n(\varepsilon'+\varepsilon^0\delta'))$, i.e. there exists $\tau\in\widehat{\Sigma}_r$ such that $\underline{i}=\underline{i}\tau$, $\underline{i}\sigma=\underline{i}\sigma'\tau$, $\varepsilon=\varepsilon'\tau$, $\varepsilon^0\delta=\varepsilon^0\delta'\tau$. Thus $\tau\in\Sigma_{\underline{i}}$, $\sigma'\in\Sigma_{\underline{i}}\sigma\tau^{-1}$, and $\delta'\tau=\delta$ since Σ_{ε^0} is trivial. Therefore $\sigma=\sigma'$, and hence $\tau\in\Sigma_{\underline{i},\underline{i}\sigma}$. So $\varepsilon=\varepsilon'$, and then $\tau\in\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon}$, and hence $\delta=\delta'$.

Now suppose $\sigma\in\Sigma_{\underline{i}}\backslash\Sigma_r/\Sigma_{\underline{i}},\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{i}\sigma},\delta\in\Sigma_{\underline{i},\varepsilon^0}\backslash\Sigma_{\underline{i}}/\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon}$ satisfy $\lambda_{\sigma,\varepsilon}\neq0$. Then by $(*)$ there exist $\sigma'\in\Sigma_{\underline{i}}\backslash\Sigma_r/\Sigma_{\underline{i}},\varepsilon'\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{i}\sigma'},\delta'\in\Sigma_{\underline{i}\sigma,\varepsilon^0\sigma}\backslash\Sigma_{\underline{i}\sigma}/\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon}$ such that $\xi_{\underline{i},\underline{i}\sigma'+n(\varepsilon'+\varepsilon^0\sigma'\delta')}=\xi_{\underline{i},\underline{i}\sigma+n(\varepsilon+\varepsilon^0\delta)}$, i.e. there exists $\tau\in\widehat{\Sigma}_r$ such that $\underline{i}=\underline{i}\tau$, $\underline{i}\sigma'=\underline{i}\sigma\tau$, $\varepsilon'=\varepsilon\tau$, $\varepsilon^0\sigma'\delta'=\varepsilon^0\delta\tau$. Similar arguments as above show that $\tau\in\Sigma_{\underline{i},\underline{i}\sigma,\varepsilon}$, and $\sigma'=\sigma$, $\varepsilon=\varepsilon'$, $\sigma\delta'=\delta\tau$. This implies $\sigma=(\sigma\delta'\sigma^{-1})^{-1}\delta\tau\in\Sigma_{\underline{i}}$, i.e. σ is trivial. In a word, $\lambda_{\sigma,\varepsilon}\neq0$ implies σ is trivial. That is, $\xi\in B_{\underline{i}}$. $\square$

Let $\lambda=(\lambda_1,\cdots,\lambda_n)$ be the weight of $\underline{i}$, i.e. $\lambda_s=\#\{k\mid k=1,\cdots,r,\ i_k=s\}$, and let α be the number of nonzero entries of λ. Then

Proposition 3.8. *$B_{\underline{i}}$ is a polynomial algebra in $r-\alpha$ indeterminates over a Laurent polynomial algebra in α indeterminates over K. In particular, $B_{\underline{i}}$ is Noetherian.*

Proof. For $s = 1, \cdots, n$, set $\underline{i}(s, r) = s \cdots s \in I(n, r)$. We may assume $\underline{i} = \underline{i}(1, \lambda_1) \cdots \underline{i}(n, \lambda_n)$. For $s = 1, \cdots, n$, and $\varepsilon \in \mathbb{Z}^r$, set

$$\Phi_s = \{0\}^{\lambda_1 + \cdots + \lambda_{s-1}} \times \mathbb{Z}^{\lambda_s} \times \{0\}^{\lambda_{s+1} + \cdots + \lambda_n},$$
$$\theta_s^0(\varepsilon) = (\varepsilon_{\lambda_1 + \cdots + \lambda_{s-1} + 1}, \cdots, \varepsilon_{\lambda_1 + \cdots + \lambda_s}) \in \mathbb{Z}^{\lambda_s},$$
$$\theta_s(\varepsilon) = (0, \cdots, 0, \theta_s^0(\varepsilon), 0, \cdots, 0) \in \Phi_s\ .$$

Let $B_s = K\{\xi_{\underline{i},\underline{i}+n\varepsilon} \mid \varepsilon \in \Phi_s\}$. If $\lambda_s = 0$ then $B_s \cong K$. Otherwise $\xi_{\underline{i},\underline{i}+n\varepsilon} \mapsto \xi_{1\cdots 1, 1\cdots 1 + \theta_s^0(\varepsilon)}$ defines an algebra isomorphism $B_s \cong \widetilde{S}(1, \lambda_s)$. By Proposition 3.6 the latter algebra is a polynomial algebra with $\lambda_s - 1$ indeterminates over a Laurent polynomial algebra in 1 indeterminates. Now for $\varepsilon \in \mathbb{Z}^r$, we have $\xi_{\underline{i},\underline{i}+n\varepsilon} = \prod_{s=1}^n \xi_{\underline{i},\underline{i}+n\theta_s(\varepsilon)}$. Therefore $B_{\underline{i}} \cong B_1 \otimes \cdots \otimes B_n$, and we are done. □

3.3 $\widetilde{S}(n, r)$ is Noetherian

In this subsection we will prove that $\widetilde{S}(n, r)$ is Noetherian.

For $\varepsilon \in \mathbb{Z}^r$, let $\breve{\varepsilon}$ be the unique element in $\{\varepsilon\sigma \mid \sigma \in \Sigma_r\}$ such that $\breve{\varepsilon}_1 \geq \breve{\varepsilon}_2 \geq \cdots \geq \breve{\varepsilon}_r$. For $\varepsilon, \varepsilon' \in \mathbb{Z}^r$, define $\varepsilon > \varepsilon'$ if $\varepsilon_1 + \cdots + \varepsilon_r = \varepsilon'_1 + \cdots + \varepsilon'_r$ and $\breve{\varepsilon} > \breve{\varepsilon}'$ according to the lexicographic order. We way $\varepsilon \in \mathbb{Z}^r$ is *successive* if $\{\varepsilon_1, \cdots, \varepsilon_r\}$ is a set of successive integers, ε is *absolutely successive* if in addition each entry is nonnegative and at least one of them equals 0.

Fix $\underline{i}, \underline{j} \in I(n, r)$. By Proposition 2.3 the K-space M spanned by $\{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \varepsilon \in \mathbb{Z}^r\}$ is a $B_{\underline{i}}$-module.

Proposition 3.9. *The module M is generated over $B_{\underline{i}}$ by*

$$\{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \varepsilon \textit{ is absolutely successive}\}\ .$$

In particular, it is finitely generated.

Proof. We prove that the basis elements $\xi_{\underline{i},\underline{j}+n\varepsilon}$ is generated by the desired set. Since

$$\xi_{\underline{i},\underline{j}+n\varepsilon} = \xi_{\underline{i},\underline{i}+nt(1,\cdots,1)} \xi_{\underline{i},\underline{j}+n(\varepsilon - t(1,\cdots,1))}$$

for $t = min\{\varepsilon_1, \cdots, \varepsilon_r\}$, we may assume $\varepsilon \geq 0$ and one of its entry is 0. Induct on the volume $\varepsilon_1 + \cdots + \varepsilon_r$ of ε.

If ε satisfies $\varepsilon_1 + \cdots + \varepsilon_r = 0$ or 1, then ε is absolutely successive. Suppose $\varepsilon_1 + \cdots + \varepsilon_r \geq 2$. If ε is successive, then it is absolutely successive. If ε is not successive, then there is a partition $\Omega \cup \Omega^c$ of $\{1, \cdots, r\}$ such that $\varepsilon_k \geq \varepsilon_{k'} + 2, \forall k \in \Omega, k' \in \Omega^c$. Recall that $e^1, \cdots, e^r$ is the natural $\mathbb{Z}$-basis for $\mathbb{Z}^r$. Set $e(\Omega) = \sum_{k \in \Omega} e^k$. Then

$$\xi_{\underline{i},\underline{j}+n\varepsilon} = \xi_{\underline{i},\underline{i}+ne_\Omega} \xi_{\underline{i},\underline{j}+n(\varepsilon - e(\Omega))} - \sum_\delta \mu_\delta \xi_{\underline{i},\underline{i}-nt_\delta(1,\cdots,1)} \xi_{\underline{i},\underline{j}+n(\varepsilon - e(\Omega) + e(\Omega)\delta - t_\delta(1,\cdots,1))}$$

where the sum is over a representative set of all nontrivial double cosets $\Sigma_{\underline{i},e(\Omega)} \delta \Sigma_{\underline{i},\varepsilon - e(\Omega),\underline{j}}$ of $\Sigma_{\underline{i}}$, $\mu_\delta = [\Sigma_{\underline{i},\underline{j},\varepsilon - e(\Omega) + e(\Omega)\delta} : \Sigma_{\underline{i},\underline{j},\varepsilon - e(\Omega), e(\Omega)\delta}]$ and t_δ is the minimal entry of $\varepsilon - e(\Omega) + e(\Omega)\delta$. Now $\varepsilon - e(\Omega)$ is of less volume. If $t_\delta = 0$ then $\varepsilon - e(\Omega) + e(\Omega)\delta$ is of the assuming form and smaller than ε, and if $t_\delta > 0$ then $\varepsilon - e(\Omega) + e(\Omega)\delta - t_\delta(1, \cdots, 1)$ is of less volume. □

As a consequence we have

Theorem 3.10. (i) *$\widetilde{S}(\underline{i})$ is a Noetherian ring.*

(ii) *$\widetilde{S}(n,r)$ is a Noetherian ring.*

Proof. By Proposition 3.9 both $\widetilde{S}(\underline{i})$ and $\widetilde{S}(n,r)$ are finitely generated over $\oplus_{\underline{i}\in I(n,r)/\Sigma_r} B_{\underline{i}}$, which is a Noetherian ring by Proposition 3.8. $\square$

In fact by a more subtle discussion we can reduce the number of generators in Proposition 3.9. We assume $\underline{i} = \underline{i}(1,\lambda_1)\cdots\underline{i}(n,\lambda_n)$. Then

Corollary 3.11. *As a $B_{\underline{i}}$-module, $K\{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \varepsilon \in \mathbb{Z}^r\}$ is generated by*

$$\{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \theta_s^0(\varepsilon) \textit{ is absolutely successive } \forall s = 1,\cdots,n\}$$

Proof. Denote by M the module under investigation. Then $M = M_1 \boxtimes \cdots \boxtimes M_n$, where $M_s = K\{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \varepsilon \in \Phi_s\}$ is a B_s-module. By Proposition 3.9, M_s is generated by $\{\xi_{\underline{i},\underline{j}+n\varepsilon} \mid \varepsilon \in \Phi_s, \theta_s^0(\varepsilon)$ is absolutely successive $\forall s = 1,\cdots,n\}$. $\square$

3.4 Special case : $\widetilde{S}(1\cdots r)$ $(n \geq r)$

Assume $n \geq r$. Let $\underline{u} = (1,\cdots,r) \in I(n,r)$. Then $\widetilde{S}(\underline{u}) = K\{\xi_{\underline{u},\underline{u}w} \mid w \in \widehat{\Sigma}_r\}$ with $\xi_{\underline{u},\underline{u}w}\xi_{\underline{u},\underline{u}w'} = \xi_{\underline{u},\underline{u}w'w}$. Note that $\xi_{\underline{u},\underline{u}w} = \xi_{\underline{u},\underline{u}w'}$ if and only if $w = w'$. Therefore the set of basis elements $\{\xi_{\underline{u},\underline{u}w} | w \in \widehat{\Sigma}_r\}$ is closed under multiplication. In fact it is isomorphic to $\widehat{\Sigma}_r$. Hence $\widetilde{S}(\underline{u})$ is isomorphic to the group algebra $K\widehat{\Sigma}_r$. We will identify these two algebras. Define *the Schur functor $F : \widetilde{S}(n,r)$-$Mod \longrightarrow K\widehat{\Sigma}_r$-$Mod$, $W \mapsto \xi_{\underline{u},\underline{u}}W$.*

Theorem 3.12. *Assume char$K > r$ or char$K = 0$. Then the Schur functor F is an equivalence.*

Proof. Since $\Sigma_{\underline{i}} \geq \Sigma_{\underline{u}}$ for any $\underline{i} \in I(n,r)$, it follows by Lemma 3.1 that each $\widetilde{S}(n,r)\xi_{\underline{i},\underline{i}}$ is a direct summand of $\widetilde{S}(n,r)\xi_{\underline{u},\underline{u}}$. Therefore $\widetilde{S}(n,r)\xi_{\underline{u},\underline{u}}$ is a progenerator, and hence we have the desired equivalence. $\square$

3.5 Special case : $\widetilde{S}(112\cdots n)$ $(r = n+1)$

Assume $r = n+1$. Let $\underline{v} = 112\cdots n$.

Theorem 3.13. *Assume char$K > n+1$ or char$K = 0$. Then $\widetilde{S}(n,n+1)$ is Morita equivalent to $\widetilde{S}(\underline{v})$.*

Proof. For any $\underline{i} \in I(n,n+1)$ there exists $\sigma \in \Sigma_{n+1}$ such that $\Sigma_{\underline{i}\sigma} \geq \Sigma_{\underline{v}}$. Therefore by Lemma 3.1 each $\widetilde{S}(n,n+1)\xi_{\underline{i},\underline{i}}$ is a direct summand of $\widetilde{S}(n,n+1)\xi_{\underline{v},\underline{v}}$. Therefore $\widetilde{S}(n,n+1)\xi_{\underline{v},\underline{v}}$ is a progenerator, and hence we have the desired Morita equivalence. $\square$

4 The center

In this section we will study the precise structure of $Z = Z(\widetilde{S}(n,r))$, the center of $\widetilde{S}(n,r)$.

For $\varepsilon \in \mathbb{Z}^r$, set

$$c_\varepsilon = \sum_{\underline{i}\in I(n,r)/\Sigma_r} \sum_{\sigma\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{i}}} \xi_{\underline{i},\underline{i}+n\varepsilon\sigma}$$

Then $c_\varepsilon = c_{\varepsilon'}$ if and only if $\varepsilon \sim_{\Sigma_r} \varepsilon'$.

Proposition 4.1. *For any $\varepsilon \in \mathbb{Z}^r$, $c_\varepsilon \in Z$.*

Proof. Let $\underline{i}, \underline{j} \in I(n,r)$, $\varepsilon^0 \in \mathbb{Z}^r$. Then by Proposition 2.3

$$\begin{aligned}
\xi_{\underline{i},\underline{j}+n\varepsilon^0} c_\varepsilon &= \xi_{\underline{i},\underline{j}+n\varepsilon^0}(\textstyle\sum_{\sigma\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{j}}} \xi_{\underline{j},\underline{j}+n\varepsilon\sigma}) = \sum_{\sigma\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{j}}} \xi_{\underline{i},\underline{j}+n\varepsilon^0}\xi_{\underline{j},\underline{j}+n\varepsilon\sigma} \\
&= \textstyle\sum_{\sigma\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{j}}} \sum_{\delta\in\Sigma_{\underline{j},\varepsilon\sigma}\backslash\Sigma_{\underline{j}}/\Sigma_{\underline{i},\underline{j},\varepsilon^0}} [\Sigma_{\underline{i},\underline{j},\varepsilon\sigma\delta+\varepsilon^0} : \Sigma_{\underline{i},\underline{j},\varepsilon^0,\varepsilon\sigma\delta}]\xi_{\underline{i},\underline{j}+n(\varepsilon\sigma\delta+\varepsilon^0)} \\
&= \textstyle\sum_{w\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{i},\underline{j},\varepsilon^0}} [\Sigma_{\underline{i},\underline{j},\varepsilon w+\varepsilon^0} : \Sigma_{\underline{i},\underline{j},\varepsilon^0,\varepsilon w}]\xi_{\underline{i},\underline{j}+n(\varepsilon w+\varepsilon^0)}
\end{aligned}$$

$$\begin{aligned}
c_\varepsilon \xi_{\underline{i},\underline{j}+n\varepsilon^0} &= (\textstyle\sum_{\sigma\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{i}}} \xi_{\underline{i},\underline{i}+n\varepsilon\sigma})\xi_{\underline{i},\underline{j}+n\varepsilon^0} = \sum_{\sigma\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{i}}} \xi_{\underline{i},\underline{i}+n\varepsilon\sigma}\xi_{\underline{i},\underline{j}+n\varepsilon^0} \\
&= \textstyle\sum_{\sigma\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{i}}} \sum_{\delta\in\Sigma_{\underline{i},\varepsilon\sigma}\backslash\Sigma_{\underline{i}}/\Sigma_{\underline{i},\underline{j},\varepsilon^0}} [\Sigma_{\underline{i},\underline{j},\varepsilon^0+\varepsilon\sigma\delta} : \Sigma_{\underline{i},\underline{j},\varepsilon\sigma\delta,\varepsilon^0}]\xi_{\underline{i},\underline{j}+n(\varepsilon^0+\varepsilon\sigma\delta)} \\
&= \textstyle\sum_{w\in\Sigma_\varepsilon\backslash\Sigma_r/\Sigma_{\underline{i},\underline{j},\varepsilon^0}} [\Sigma_{\underline{i},\underline{j},\varepsilon w+\varepsilon^0} : \Sigma_{\underline{i},\underline{j},\varepsilon^0,\varepsilon w}]\xi_{\underline{i},\underline{j}+n(\varepsilon w+\varepsilon^0)}
\end{aligned}$$

The last equalities in these two formulas follow from the following lemma. $\square$

Lemma 4.2. *Let G be a group, H_1, H_2 are two subgroups of G and H_3 a subgroup of H_2. Let Σ be a representative set of $H_1\backslash G/H_2$, and for $\sigma \in \Sigma$ let $\eta(\sigma)$ be a representative set of $(H_1^\sigma \cap H_2)\backslash H_2/H_3$, where $H_1^\sigma = \sigma^{-1}H_1\sigma$. Then $\cup_{\sigma\in\Sigma}\sigma\eta(\sigma)$ is a representative set of $H_1\backslash G/H_3$.*

Proposition 4.3. *Z is spanned by $\{c_\varepsilon \mid \varepsilon \in \mathbb{Z}^r\}$.*

Before proving this proposition, we first have a look at what form a central element should have. Let $c = \sum_{\underline{i}\in I(n,r)/\Sigma_r} \sum_{\underline{j}\in I(n,r)/\Sigma_{\underline{i}}} \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i},\underline{j}}} \lambda_{\underline{i},\underline{j},\varepsilon}\xi_{\underline{i},\underline{j}+n\varepsilon} \in \widetilde{S}(n,r)$ be a central element. Then for any $\underline{i} \in I(n,r)/\Sigma_r$, we have $\xi_{\underline{i},\underline{i}}c \in \widetilde{S}(\underline{i})$. Moreover $\xi_{\underline{i},\underline{i}}c$ lies in the center of $\widetilde{S}(\underline{i})$, and hence lies in $B_{\underline{i}}$ by Proposition 3.7. Therefore $\lambda_{\underline{i},\underline{j},\varepsilon} = 0$ if $\underline{j} \neq \underline{i}$. Namely c can be written as

$$c = \sum_{\underline{i}\in I(n,r)/\Sigma_r} \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i}}} \lambda_{\underline{i},\varepsilon}\xi_{\underline{i},\underline{i}+n\varepsilon}$$

Lemma 4.4. *Let $\underline{i} \in I(n,r)$ and assume $\xi \in B_{\underline{i}}$ satisfies $\xi_{1\cdots1,\underline{i}}\xi = 0$. Then $\xi = 0$.*

Proof. Suppse $\xi = \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i}}} \lambda_\varepsilon\xi_{\underline{i},\underline{i}+n\varepsilon} \in B_{\underline{i}}$. Then $\xi_{1\cdots1,\underline{i}}\xi = \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i}}} \lambda_\varepsilon\xi_{1\cdots1,\underline{i}+n\varepsilon}$. But for $\varepsilon, \varepsilon' \in \mathbb{Z}^r/\Sigma_{\underline{i}}$, we have $\xi_{1\cdots1,\underline{i}+n\varepsilon} = \xi_{1\cdots1,\underline{i}+n\varepsilon'}$ if and only if $\varepsilon = \varepsilon'$. Therefore $\xi_{1\cdots1,\underline{i}}\xi = 0$ implies $\lambda_\varepsilon = 0$ for any $\varepsilon \in \mathbb{Z}^r/\Sigma_{\underline{i}}$. $\square$

Now we are ready for the

Proof of Proposition 4.3 :

Let $c = \sum_{\underline{i}\in I(n,r)/\Sigma_r} \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i}}} \lambda_{\underline{i},\varepsilon}\xi_{\underline{i},\underline{i}+n\varepsilon} \in \widetilde{S}(n,r)$ be a central element. Then $\xi = c - \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i}}} \lambda_{1\cdots1,\varepsilon}c_\varepsilon$ is also a central element. Moreover $\xi_{1\cdots1,\underline{i}}\xi\xi_{\underline{i},\underline{i}} = \xi_{1\cdots1,\underline{i}}\xi = \xi\xi_{1\cdots1,\underline{i}} = 0$ for any $\underline{i} \in I(n,r)$. It follows from Lemma 4.4 that $\xi\xi_{\underline{i},\underline{i}} = 0$ for any $\underline{i} \in I(n,r)$, and hence $\xi = 0$. Therefore $c = \sum_{\varepsilon\in\mathbb{Z}^r/\Sigma_{\underline{i}}} \lambda_{1\cdots1,\varepsilon}c_\varepsilon$, as desired. $\square$

Theorem 4.5. *The center Z of $\widetilde{S}(n,r)$ is isomorphic to $K[t_1, \cdots, t_{r-1}, t_r, t_r^{-1}]$, where $t_1, \cdots, t_r$ are indeterminates. To be precise, $Z = K[c_{\varepsilon^1}, \cdots, c_{\varepsilon^{r-1}}, c_{\varepsilon^r}, c_{-\varepsilon^r}]$, where $\varepsilon^1, \cdots, \varepsilon^r$ are defined in Section 3.1. In particular, $\widetilde{S}(n,r)$ is indecomposable.*

Proof. Sending c_ε to $c_\varepsilon\xi_{1\cdots1,1\cdots1}$ defines a K-algebra isomorphism from Z to $\widetilde{S}(1\cdots1)$. $\square$

5 Examples

We shall denote by $\mathcal{M}_n$ the algebra of $n \times n$-matrices with entries from the field K.

Example 1. Let $r = 1$. Then $\widetilde{S}(n,1)$ is isomorphic to $\mathcal{M}_n \otimes K[t,t^{-1}]$. So $\widetilde{S}(n,1)$ is Morita equivalent to $K[t,t^{-1}] = K\widehat{\Sigma}_1$ with the equivalence given by the Schur functor. The center of this algebra is $1 \otimes K[t,t^{-1}]$.

Example 2. Let $n = 1$. Then by Proposition 3.6 $\widetilde{S}(1,r) \cong K[t_1,\cdots,t_{r-1},t_r,t_r^{-1}]$, where $t_1,\cdots,t_r$ are indeterminates. This is a commutative algebra.

Example 3. Let $n = 2$, $r = 2$. Then by Theorem 4.5 the center of $\widetilde{S}(2,2)$ is $Z = K[c_{10},c_{11},c_{11}^{-1}]$. We shall classify all simple $\widetilde{S}(2,2)$-modules. Let M be a simple $\widetilde{S}(2,2)$-module, then Z acts as scalars, say, c_{10} as $a \in K$, and c_{11} as $b \in K^{\times}$. Then the action of $\widetilde{S}(2,2)$ on M factors through the algebra $A_{a,b} = \widetilde{S}(2,2)/(c_{10}-a, c_{11}-b)$.

Case 1 : $charK \neq 2$. As a Z-module, $\widetilde{S}(2,2)$ is free with basis

$$\begin{array}{cccc}
\xi_{11,11} & \xi_{11,12} & \xi_{11,12}\gamma & \xi_{11,22} \\
\frac{\xi_{12,11}}{2} & \frac{\xi_{12,12}+\xi_{12,21}}{2} & \gamma & \frac{\xi_{12,22}}{2} \\
\frac{\gamma'\xi_{12,11}}{2} & \gamma' & \frac{\xi_{12,12}-\xi_{12,21}}{2} & \frac{\gamma'\xi_{12,22}}{2} \\
\xi_{22,11} & \xi_{22,12} & \xi_{22,12}\gamma & \xi_{22,22}
\end{array}$$

where $\frac{\xi_{12,12}-\xi_{12,21}}{2}\,\gamma'\,\frac{\xi_{12,12}+\xi_{12,21}}{2} = \gamma'$, $\frac{\xi_{12,12}+\xi_{12,21}}{2}\,\gamma\,\frac{\xi_{12,12}-\xi_{12,21}}{2} = \gamma$, $\gamma'\gamma = (c_{10}^2 - 4c_{11})\frac{\xi_{12,12}-\xi_{12,21}}{2}$ and $\gamma\gamma' = (c_{10}^2 - 4c_{11})\frac{\xi_{12,12}+\xi_{12,21}}{2}$. Other structure constants are easy to calculate.

When $a^2 - 4b = 0$, the radical $radA_{a,b}$ of $A_{a,b}$ is spanned by $\xi_{11,12}\gamma$, γ, $\frac{\gamma'\xi_{12,11}}{2}$, γ', $\frac{\gamma'\xi_{12,22}}{2}$, and $\xi_{22,12}\gamma$, and hence $A_{a,b}/rad(A_{a,b})$ is isomorphic to $\mathcal{M}_3 \times K$. When $a^2 - 4b \neq 0$, the algebra $A_{a,b}$ is isomorphic to $\mathcal{M}_4$.

Therefore the isoclasses of simple $\widetilde{S}(2,2)$-modules are parametrized by the plane without the x-axis and with the curve $\{(x, \frac{x^2}{4}) \mid x \neq 0\}$ doubled.

Case 2 : $charK = 2$. As a Z-module, $\widetilde{S}(2,2)$ is free with basis

$$\begin{array}{cccc}
\xi_{11,11} & \xi_{11,12} & \xi_{11,12}l & \xi_{11,22} \\
\xi_{12,11} & \xi_{12,12} & \xi_{12,21} & \xi_{12,22} \\
l\xi_{12,11} & l & l\xi_{12,21} & l\xi_{12,22} \\
\xi_{22,11} & \xi_{22,12} & \xi_{22,12}l & \xi_{22,22}
\end{array}$$

where $\xi_{12,12}l = l\xi_{12,12} = l$, $l^2 = c_{10}l + c_{11}\xi_{12,12}$, $\xi_{12,21}l = l\xi_{12,21} + c_{10}\xi_{12,21}$, $\xi_{11,12}l\xi_{12,11} = c_{10}\xi_{11,11}$, $\xi_{11,12}l\xi_{12,22} = c_{10}\xi_{11,22}$, $\xi_{22,12}l\xi_{12,11} = c_{10}\xi_{22,11}$, and $\xi_{22,12}l\xi_{22,22} = c_{10}\xi_{12,22}$. Other structure constants are easy to calculate.

We denote the images of these elements in $A_{a,b}$ by the same notations. For simplicity we assume K is quadratically closed. Let $\alpha = \xi_{12,12} + \xi_{12,21}$, $\beta = l\xi_{12,21} + \sqrt{b}\xi_{12,12}$, then $A_{a,b}$ has the following basis

$$\begin{array}{cccc}
\xi_{11,11} & \xi_{11,12} & \xi_{11,12}\beta & \xi_{11,22} \\
\xi_{12,11} & \xi_{12,12} & \alpha & \xi_{12,22} \\
\beta\xi_{12,11} & \beta & \alpha\beta & \beta\xi_{12,22} \\
\xi_{22,11} & \xi_{22,12} & \xi_{22,12}\beta & \xi_{22,22}
\end{array}$$

where $\beta^2 = 0$, $\beta\alpha = \alpha\beta + a$, $\xi_{12,12}\beta\xi_{12,12} = \beta$, $\xi_{\underline{i},12}\beta\xi_{12,\underline{j}} = a\xi_{\underline{i},\underline{j}}$, $\xi_{12,\underline{i}}\xi_{\underline{i},12} = \alpha$, $\xi_{\underline{i},12}\xi_{12,\underline{j}} = 0$. In the above, $\underline{i},\underline{j} \in \{11,22\}$.

If $a = 0$, then the radical $radA_{0,b}$ of $A_{0,b}$ is spanned by $\xi_{11,12}$, $\xi_{11,12}\beta$, $\xi_{12,11}$, α, $\xi_{12,22}$, $\beta\xi_{12,11}$, β, $\alpha\beta$, $\beta\xi_{12,22}$, $\xi_{22,12}$, $\xi_{22,12}\beta$. Therefore $A_{0,b}/radA_{0,b}$ is isomorphic to $\mathcal{M}_2 \times K$. If $a \neq 0$, then

the algebra $A_{a,b}$ has basis

$$\begin{array}{cccc}
\xi_{11,11} & \xi_{11,12} & \xi_{11,12}\beta & \xi_{11,22} \\
a^{-1}\beta\xi_{12,11} & 1+a^{-1}\alpha\beta & \beta & a^{-1}\beta\xi_{12,22} \\
a^{-1}\xi_{12,11} & a^{-1}\alpha & a^{-1}\alpha\beta & a^{-1}\xi_{12,22} \\
\xi_{22,11} & \xi_{22,12} & \xi_{22,12}\beta & \xi_{22,22}
\end{array}$$

If we denote by E_{ij} the element in the (i,j)-entry then $E_{ij}E_{kl} = \delta_{jk}E_{il}$. In particular $A_{a,b}$ is isomorphic to $\mathcal{M}_4$.

Therefore the isoclasses of simple $\widetilde{S}(2,2)$-modules are parametrized by the plane without x-axis and with the curve $\{(0,y) \mid y \neq 0\}$ doubled.

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